MEM6810 Engineering Systems Modeling and Simulation
工程系统建模与仿真

Theory Analysis

Lecture 3: Queueing Models

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Contents

- Queues (or waiting lines) are EVERYWHERE!
- Queues are an unavoidable component of modern life.
  - E.g., in hospital, stores, bank, call center (online service), etc.
  - Although we don't like standing in a queue, we appreciate the fairness that it imposes.
- Queues are not just for humans, however.
  - E.g., email system, printer, manufacturing line, etc.
  - Manufacturing systems maintain queues (called inventories) of raw materials, partly finished goods, and finished goods via the manufacturing process.

Figure: Queues in Hospital

Figure: Queues in Store (*from* *The Sun*)

Figure: Queues in Campus (for COVID-19 Nucleic Acid Test)

Figure: Queues in Bank

Figure: Queues in Bank (No requirement to *stand physically* in queues)

Figure: Queue in Online Service

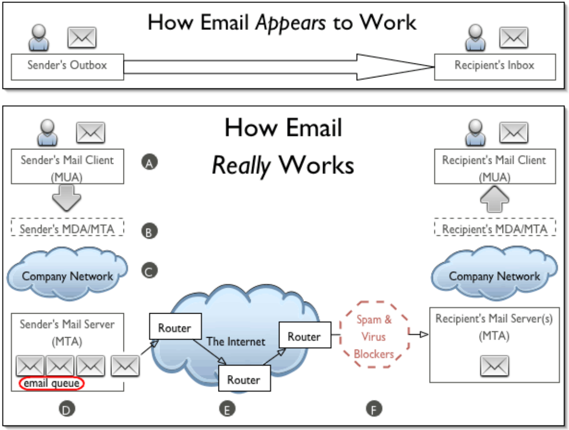


Figure: Queue in Mail Server (*from* OASIS)

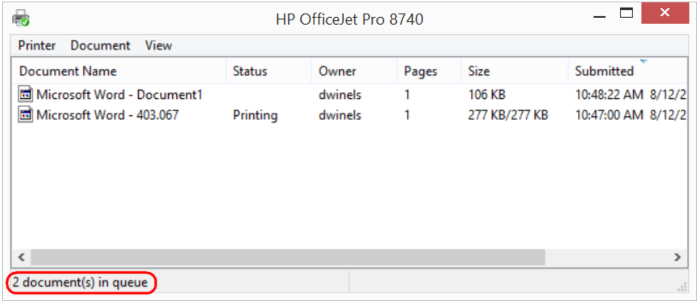


Figure: Queue in Printer

Figure: Queues (Inventories) in Manufacturing Line (*from* *Estes*)

- Typically, a queueing system consists of a stream of "**customers**" (humans, goods, messages) that
  - arrive at a service facility;
  - wait in the **queue** according to certain discipline;
  - get served;
  - finally depart.
- A lot of real-world systems can be viewed as queueing systems, e.g.,
  - service facilities
  - production systems
  - repair and maintenance facilities
  - communications and computer systems
  - transport and material-handling systems, etc.
- Queueing models are mathematical representation of queueing systems.

- Queueing models may be
  - *analytically solved using queueing theory* when they are simple (highly simplified); or
  - *analyzed through simulation* when they are complex (more realistic).

- Studied in either way, queueing models provide us a powerful tool for designing and evaluating the performance of queueing systems.

- They help us do this by answering the following questions (and many others):
  1. How many customers are there in the queue (or station) on average?
  2. How long does a typical customer spend in the queue (or station) on average?
  3. How busy are the servers on average?

- *Simple queueing models solved analytically*:
  - Get rough estimates of system performance with negligible time and expense.
  - *More importantly, understand the dynamic behavior of the queueing systems and the relationships between various performance measures.*
  - Provide a way to verify that the simulation model has been programmed correctly.
- *Complex queueing models analyzed through simulation*:
  - Allow us to incorporate arbitrarily fine details of the system into the model.
  - Estimate virtually any performance measure of interest with high accuracy.
- This lecture focuses on the classical analytically solvable queueing models.

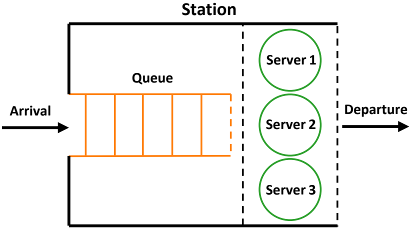
- The key elements of a queueing system are the **customers** and **servers**.
  - The term customer can refer to anything that arrives and requires service.
  - The term server can refer to any resource that provides the requested service.

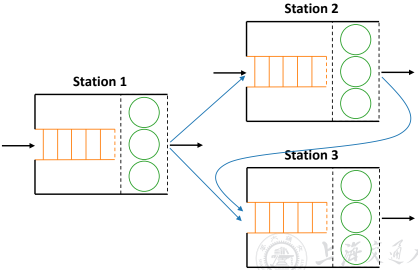
- The term **station** means the entire or part of the system, which contains all the identical servers and the queue.

- Suppose that there is only **one queue** in one station.

- **Capacity** is the maximal number of customers allowed in the station.
  - Number waiting in queue + number having service.
  - Finite or infinite.

- Single-station queueing system.
  - Customers simply leave after service.
  - E.g., customers arrive to buy coffee and then leave.
- Multiple-station queueing system (queueing network).
  - Customers can move from one station to another (for different service), before leaving the system.
  - E.g., patients wait and get service at several different units inside a hospital.





- The **arrival process** describes how the customers come.
  - Arrivals may occur at *scheduled* times or *random* times.
  - When at random times, the **interarrival times** are usually characterized by a probability distribution.
  - Customers may arrive one at a time or in batch (with constant or random batch size).
  - Different types of customers.

- An customer arriving at a station:
  - if the station capacity is full:
    - the external arrival will leave immediately (called **lost**);
    - the internal arrival may wait in the previous station (may **block** the previous server).
  - if the station capacity is not full, enter the station:
    - if there is idle server in the station, get service immediately;
    - if all servers are busy, wait in the **queue**.

- Queue discipline: Which customer to serve first.
  - First-in-first-out (FIFO), or first-come-first-served (FCFS).
  - Last-in-first-out (LIFO), or last-come-first-served (LCFS).
  - Shortest processing time first.
  - Service according to priority (more than one customer types).
- Queue behavior: Actions of customers while waiting.
  - Balk: leave when they see that the line is too long.
  - Renege: leave after being in the line when they see that the line is moving too slowly.
- **Service time** is the duration of service in a server.
  - *Constant* or *random* duration.
  - May depend on the customer type.
  - May depend on the time of day or the queue length.

- When without specification, the queueing models considered in this lecture shall satisfy the following:
  1. One customer type.
  2. Random arrivals (i.e., random interarrival times, iid.).
  3. No batch (or say, batch size is 1).†
  4. One queue in one station.
  5. First-come-first-served (FCFS).
  6. No balk, no renege.
  7. Random service time (depends on nothing else), iid.

- Even so, it is not that easy to analyze the queueing models!

† 1+2+3 ⇒ The arrival process is a *renewal process*.

- Canonical notational system proposed by Kendall (1953): $X/Y/s/K$.
    - X represents the interarrival-time distribution.
        - M: Memoryless, i.e., exponential interarrival times;
        - G: General;
        - D: Deterministic.
    - Y represents the service-time distribution.
        - Same letters as the interarrival times.
    - s represents the number of parallel servers.
        - Finite value.
        - For infinite number of servers, s is replaced by ∞.
    - K represents the station capacity.
        - Finite value.
        - For infinite capacity, K is replaced by ∞, or simply omitted.
- Examples: $M/M/1$, $M/G/1$, $M/M/s/K$.

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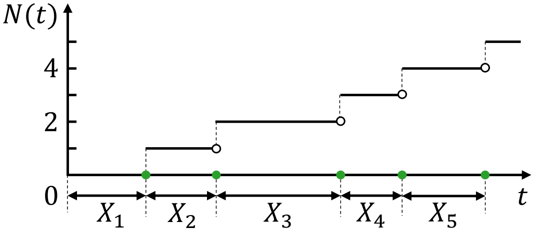
- A stochastic process $\{N(t),\ t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of arrivals that have occurred up to time t.



- Let $\{X_n,\ n \geq 1\}$ denote the *interarrival times*:
  - X_1 denotes the time of the first arrival;
  - For $n \geq 2$, X_n denotes the time between the $(n-1)$st and the nth arrivals.

- **Definition 1**. The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has **independent** and **stationary** increments;
  - For $t > 0$, $N(t) \sim \mathrm{Pois}(\lambda t)$, i.e.,

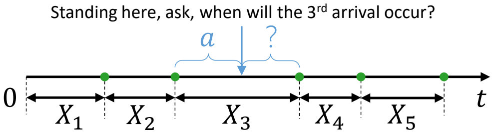
$$\mathbb{P}(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots.$$

- Independent Increments: The numbers of arrivals in disjoint time intervals are independent.
- Stationary Increments: The distribution of number of arrivals in any time interval depends only on the length of time interval, i.e., for $s < t$, the distribution of $N(t) - N(s)$ depends only on $t - s$.

Poisson process $\subset$ renewal process $\subset$ counting process.

- **Definition 2**. The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - The process has independent and stationary increments;
  - $\mathbb{P}(N(t) = 1) = \lambda t + o(t)$;
  - $\mathbb{P}(N(t) \geq 2) = o(t)$.
- **Definition 3**. The counting process $\{N(t),\ t \geq 0\}$ is called a **Poisson process** with rate λ, $\lambda > 0$, if:
  - $N(0) = 0$;
  - $\{X_n,\ n \geq 1\}$ is a sequence of iid random variables, and $X_n \sim \mathrm{Exp}(\lambda)$.
- **Definition 1**, **Definition 2** and **Definition 3** are equivalent.

- **Question 1**: When will the next appear?

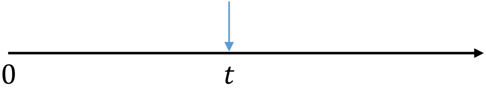


$$\begin{aligned}
\mathbb{P}(X_3 - a > x | X_3 > a) &= \frac{\mathbb{P}(X_3 - a > x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x, X_3 > a)}{\mathbb{P}(X_3 > a)} \\
&= \frac{\mathbb{P}(X_3 > a + x)}{\mathbb{P}(X_3 > a)} \\
&= \frac{e^{-\lambda(a+x)}}{e^{-\lambda a}} = e^{-\lambda x}. \quad \text{(Not related to } a\text{!)}
\end{aligned}$$

- The Poisson process has no memory! (equivalent to the independent and stationary increments assumption)

- Let $S_n = X_1 + X_2 + \cdots + X_n$ be the arrival time of the nth arrival.
- **Question 2**: If I only know there are n arrivals up to time t, what can I say about the n arrival times $S_1, \ldots, S_n$?
- A simplified case:

I'm only told that up to time t, one arrival has occurred.
What is the distribution of that arrival time S_1 ($= X_1$) ?



- Intuition:
  - Since Poisson process possesses independent and stationary increments, each interval of equal length in $[0, t]$ should have the same probability of containing the arrival.
  - Hence, the arrival time should be uniformly distributed on $[0, t]$.

Proof.

$$
\begin{aligned}
\mathbb{P}\{X_1 < s | N(t) = 1\} &= \frac{\mathbb{P}\{X_1 < s, N(t) = 1\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\mathbb{P}\{1 \text{ arrival in } [0, s), 0 \text{ arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \\
&= \frac{\mathbb{P}\{1 \text{ arrival in } [0, s)\}\,\mathbb{P}\{0 \text{ arrival in } [s, t)\}}{\mathbb{P}\{N(t) = 1\}} \quad \text{(independent)} \\
&= \frac{\mathbb{P}\{N(s) = 1\}\,\mathbb{P}\{N(t-s) = 0\}}{\mathbb{P}\{N(t) = 1\}} \quad \text{(stationary)} \\
&= \frac{e^{-\lambda s}\lambda s e^{-\lambda(t-s)}}{e^{-\lambda t}\lambda t} \\
&= \frac{s}{t}.
\end{aligned}
$$

■

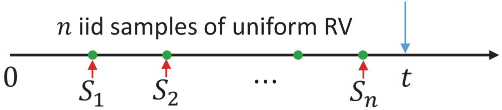
- Remark: This result can be generalized to n arrivals.

Property (Conditional Distribution of Arrival Times)

Given that $N(t) = n$, the n arrival times $S_1, \ldots, S_n$ have the same distribution as the order statistics corresponding to n independent RVs uniformly distributed on the interval $(0, t)$.

- Illustration:

Given $N(t) = n$, how can I generate a sample of $\{S_1, S_2, \ldots, S_n\}$?



1. **Uniformly** and **independently** sample n points on $[0, t]$.
2. From small to large, call them $S_1, S_2, \ldots, S_n$.

- This is very nice for simulation!

- Let $L(t)$ denote the number of customers in the station at time t.

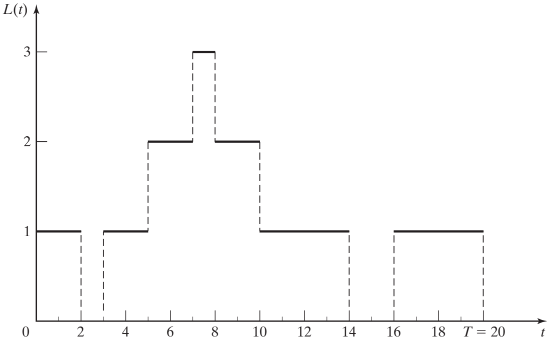


Figure: Illustration of $L(t)$ (*from Banks et al. (2010)*)

- Let $\widehat{L}(T)$ denote the (time-weighted) average number of customers in the station up to time T:

$$\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t.$$

- Another expression of $\widehat{L}(T)$: Let T_n denote the total time during $[0, T]$ in which the station contains exactly n customers.

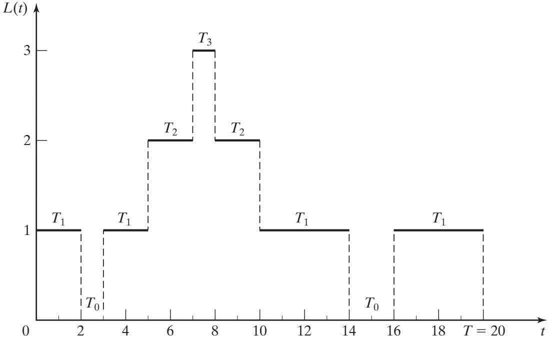


Figure: Illustration of $L(t)$ (*from* *Banks et al. (2010)*)

- $\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right).$

- Suppose during time $[0, T]$, totally $N(T)$ customers have entered the station, and let $W_1, W_2, \ldots, W_{N(T)}$ denote the time each customer spends in the station up to time T.†
- Let $\widehat{W}(T)$ denote the average sojourn time (逗留时间) in the station up to time T:

$$\widehat{W}(T) := \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i.$$

- In a similar way, we can also define
  - $\widehat{L}_Q(T)$ – The average number of customers in the *queue* up to time T.
  - $\widehat{W}_Q(T)$ – The average *waiting* time in the *queue* up to time T.

† The time includes both the waiting time in queue and the time in server. The part after T is not counted.

- Now we consider the long-run measures.
  - L – The long-run average number of customers in the station:
  $$L := \lim_{T\to\infty} \widehat{L}(T).$$
  - W – The long-run average sojourn time in the station:
  $$W := \lim_{T\to\infty} \widehat{W}(T).$$
  - L_Q – The long-run average number of customers in the queue:
  $$L_Q := \lim_{T\to\infty} \widehat{L}_Q(T).$$
  - W_Q – The long-run average waiting time in the queue:
  $$W_Q := \lim_{T\to\infty} \widehat{W}_Q(T).$$
- Question: When will L, W, L_Q and W_Q exist (and $< \infty$)?

- We also define the *limiting probability* that there will be exactly n customers in the station as time goes to infinity:

$$P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \quad n = 0, 1, 2, \ldots.$$

- Question: When will P_n exist?

- Moreover, for an arbitrary $X/Y/s/K$ queue
  - Let λ denote the arrival rate, i.e.,

$$\mathbb{E}[\text{interarrival time}] = \frac{1}{\lambda}.$$

  - Let μ denote the service rate in one server, i.e.,

$$\mathbb{E}[\text{service time}] = \frac{1}{\mu}.$$

- Question: When will L, W, L_Q, W_Q and P_n exist?
- Answer: When the queue is **stable**.†
- Question: When will the queue be stable?!

Theorem 1 (Condition of Stability)

For an $X/Y/s/\infty$ queue (i.e., infinite capacity) with arrival rate λ and service rate μ, it is stable if

$$\lambda < s\mu.$$

And, an $X/Y/s/K$ queue (i.e., finite capacity) will always be stable.

† That is to say, the underlying Markov chain is positive recurrent.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\},\ n = 0, 1, 2, \ldots$.
- P_n is also called the probability that there are exactly n customers in the station when it is in the *steady state*.
  - Since the system is stable and run for infinitely long time, it should enters some steady state (i.e., has nothing to do with the initial state).
- L can also be written as $L := \sum_{n=0}^{\infty} nP_n$ (see next slide).
  - L is also called the expected number of customers in the station in steady state;
  - W is also called the expected sojourn time in the station in steady state;
  - L_Q is also called the expected number of customers in the queue in steady state;
  - W_Q is also called the expected waiting time in the queue in steady state.

- Recall that $P_n := \lim_{t\to\infty} \mathbb{P}\{L(t) = n\}, \ n = 0, 1, 2, \ldots$.
- It turns out that, when the queue is *stable*, P_n also equals the *long-run proportion of time that the station contains exactly* n *customers*,[†] i.e., with probability 1, for all n,

$$P_n = \lim_{T\to\infty} \frac{\text{amount of time during } [0,T] \text{ that station contains } n \text{ customers}}{T}.$$

- Recall $\widehat{L}(T) := \frac{1}{T}\int_0^T L(t)\mathrm{d}t = \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right)$, then

$$\begin{aligned} L := \lim_{T\to\infty} \widehat{L}(T) &= \lim_{T\to\infty} \sum_{n=0}^{\infty} n\left(\frac{T_n}{T}\right) \\ &= \sum_{n=0}^{\infty} \lim_{T\to\infty} n\left(\frac{T_n}{T}\right) \qquad \text{(by DCT)} \\ &= \sum_{n=0}^{\infty} nP_n. \end{aligned}$$

[†] A sufficient condition is that the queueing process is regenerative, which is satisfied in our discussion.

- Little's Law (守恒方程) is one of the most general and versatile laws in queueing theory.
  - It is named after John D.C. Little, who was the first to prove a version of it, in 1961.
  - When used in clever ways, Little's Law can lead to remarkably simple derivations.

Theorem 2 (Little's Law – Empirical Version)

Define the observed entering rate $\widehat{\lambda} := N(T)/T$, then

$$\widehat{L}(T) = \widehat{\lambda}\widehat{W}(T), \quad \widehat{L}_Q(T) = \widehat{\lambda}\widehat{W}_Q(T).$$

- Verify Little's Law.

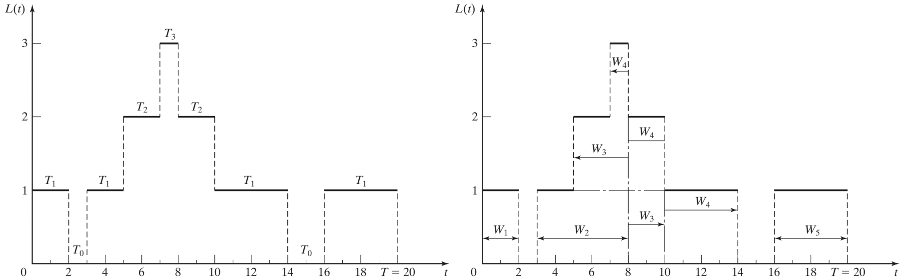


Figure: Illustration of $L(t)$ and W_i (*from Banks et al. (2010)*)

$$\widehat{\lambda} = N(T)/T = 5/20 = 0.25.$$

$$\widehat{W}(T) = \frac{1}{N(T)} \sum_{i=1}^{N(T)} W_i = \frac{1}{5}(2+5+5+7+4) = \frac{23}{5} = 4.6.$$

$$\widehat{L}(T) = \frac{1}{T} \sum_{n=0}^{\infty} nT_n = \frac{1}{20}(0 \times 3 + 1 \times 12 + 2 \times 4 + 3 \times 1) = \frac{23}{20} = 1.15.$$

So, $\widehat{\lambda}\widehat{W}(T) = 0.25 \times 4.6 = 1.15 = \widehat{L}(T)$. **(Why it always holds?)**

- Verify Little's Law.

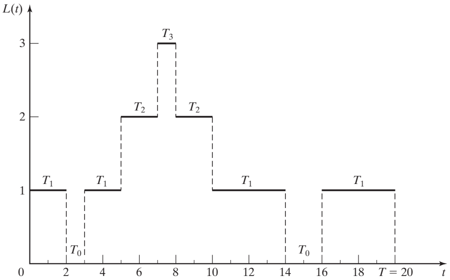


Figure: Illustration of $L(t)$ and W_i (*from Banks et al. (2010)*)

- **Why it always holds?**

$\widehat{L}(T) = \frac{1}{T}\sum_{n=0}^{\infty} nT_n = \frac{1}{T} \times$ area.

$\widehat{\lambda}\widehat{W}(T) = \frac{N(T)}{T}\frac{1}{N(T)}\sum_{i=1}^{N(T)} W_i = \frac{1}{T}\sum_{i=1}^{N(T)} W_i = \frac{1}{T} \times$ area.

So, $\widehat{L}(T) = \widehat{\lambda}\widehat{W}(T)$ always holds.

- **The same argument for $\widehat{L}_Q(T) = \widehat{\lambda}\widehat{W}_Q(T)$.**

Theorem 3 (Little's Law – Limit/Expectation Version)

For a stable queue, let λ^* denote the arrival rate or entering rate, then

$$L = \lambda^* W, \quad L_Q = \lambda^* W_Q.$$

Caution: When λ^* is the arrival rate, the time average (W, W_Q) is based on all customers (who enter the station or are lost); When λ^* is the entering rate, the time average is only based on the customers who enters the station.

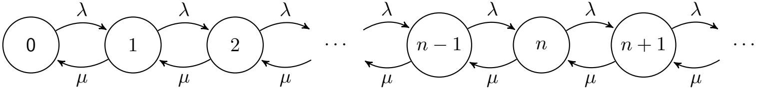
- Some Remarks:
  - For a customer who is lost (due to the finite capacity), he spends 0 amount of time in the station (or queue).
  - Once we know anyone of L, W, L_Q and W_Q, we can compute the rest using Little's Law.

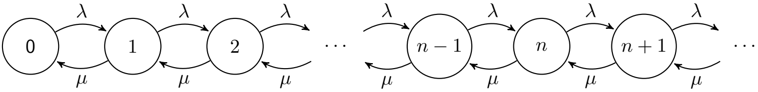
- $M/M/1$ Queue†
  - The interarrival times are iid random variables with $\mathrm{Exp}(\lambda)$ distribution, that is to say, *customers arrive according to a Poisson process with rate* λ.
  - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/M/1$ queue is stable **if and only if** $\lambda < \mu$.
  - Due to unlimited capacity, arrival rate $=$ entering rate.

- We now want to compute all the measures P_n, L, W, L_Q and W_Q.

† $M/M/1$ Queue $\subset$ *Birth and Death Process with Infinite Capacity* $\subset$ Continuous-Time Markov Chain.

- Recall that L can be computed via $L = \sum_{n=0}^{\infty} nP_n$, where P_n has two interpretations:
  - Long-run proportion of time that the station contains exactly n customers;
  - Probability that there are exactly n customers in the station as time goes to infinity (or equivalently, in the steady state).
- Define the **state** as the the number of customers in the system.
- The state space diagram is as follows:



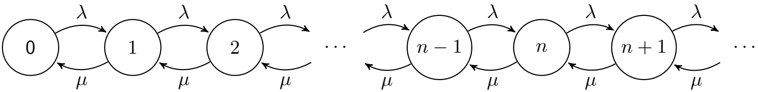


Key Observation 1

Rate at which the process leaves state n
$=$ Rate at which the process enters state n.

Heuristic Proof.

- In any time interval, the number of transitions into state n must equal to within 1 the number of transitions out of state n. (Why?)
- Hence, in the long run, the rate into state n must equal the rate out of state n.



Key Observation 2

Rate at which the process leaves state $0 = P_0\lambda$;
Rate at which the process leaves state $n = P_n(\mu+\lambda)$, $n \geq 1$;
Rate at which the process enters state $0 = P_1\mu$;
Rate at which the process enters state $n = P_{n-1}\lambda + P_{n+1}\mu$, $n \geq 1$.

Fact

If $X_1, \ldots, X_n$ are independent random variables, and $X_i \sim \mathrm{Exp}(\lambda_i)$, $i = 1, \ldots, n$, then

$$\min\{X_1, \ldots, X_n\} \sim \mathrm{Exp}(\lambda_1 + \cdots + \lambda_n).$$

Theorem 4 (Limiting Distribution of $M/M/1$ Queue)

For an $M/M/1$ queue, when it is stable ($\lambda < \mu$), its limiting (steady-state) distribution is given by

$$P_n = (1-\rho)\rho^n, \quad n \geq 0,$$

where $\rho := \lambda/\mu < 1$. (ρ is called the *server utilization*.)

Proof. Due to Observations 1 & 2,

State	Rate Process Leaves		Rate Process Enters
0	$P_0\lambda$	$=$	$P_1\mu$
n, $n \geq 1$	$P_n(\mu+\lambda)$	$=$	$P_{n-1}\lambda + P_{n+1}\mu$

Rewriting these equations gives

$$P_0\lambda = P_1\mu,$$
$$P_n\lambda = P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1.$$

Recall that

$$\begin{aligned} P_0\lambda &= P_1\mu, \\ P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad n \geq 1. \end{aligned}$$

Or, equivalently,

$$\begin{aligned} P_0\lambda &= P_1\mu, \\ P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\ P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\ P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad n \geq 1. \end{aligned}$$

Let $\rho \coloneqq \lambda/\mu$ (< 1), solving in terms of P_0 yields

$$\begin{aligned} P_1 &= P_0\rho, \\ P_2 &= P_1\rho = P_0\rho^2, \\ P_n &= P_{n-1}\rho = P_0\rho^n, \quad n \geq 1. \end{aligned}$$

Since $1 = \Sigma_{n=0}^{\infty} P_n = P_0 \Sigma_{n=0}^{\infty} \rho^n = P_0/(1-\rho)$, we have

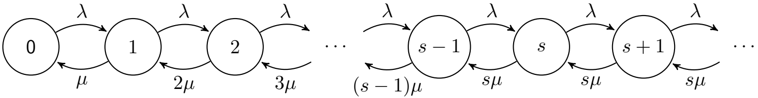
$$P_0 = 1 - \rho, \quad \text{and} \quad P_n = (1-\rho)\rho^n, \quad n \geq 1. \qquad \blacksquare$$

- $L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = \frac{\rho}{1-\rho}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$.
- $L_Q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty}(n-1)(1-\rho)\rho^n = \frac{\rho^2}{1-\rho}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{\rho^2}{1-\rho} = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{\rho}{\mu-\lambda}$.
- Or, $W_Q = W - \mathbb{E}[\text{service time}] = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu-\lambda}$.
- Using Little's Law, $L_Q = \lambda W_Q = \lambda\frac{\rho}{\mu-\lambda} = \frac{\rho^2}{1-\rho}$.
- Due to unlimited capacity, arrival rate $=$ entering rate, so the time average (W, W_Q) is based on all customers.
- $\mathbb{P}(\text{the server is idle}) = P_0 = 1-\rho$.
- As $\rho \to 1$, all L, W, L_Q and W_Q tend to ∞.

- $M/M/s$ Queue†
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
  - There are s parallel servers.
  - The customers form a single queue and get served by the next available server in an FCFS fashion.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/M/s$ queue is stable **if and only if** $\lambda < s\mu$.
  - Due to unlimited capacity, arrival rate $=$ entering rate.
- $M/M/s$ queue is a generalized version of $M/M/1$ queue. Let $s = 1$, all results should degenerate to those of $M/M/1$.

† $M/M/1$ Queue $\subset$ $M/M/s$ Queue $\subset$ *Birth and Death Process with Infinite Capacity* $\subset$ CTMC.

- The state space diagram is as follows:



Theorem 5 (Limiting Distribution of $M/M/s$ Queue)

For an $M/M/s$ queue, when it is stable ($\lambda < s\mu$), its limiting (steady-state) distribution is given by

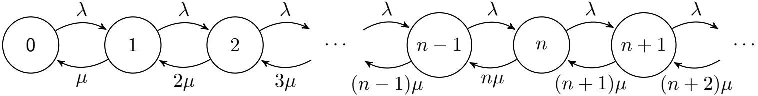
$$P_n = \left[\sum_{i=0}^{s} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \frac{s^s}{s!}\frac{\rho^{s+1}}{1-\rho}\right]^{-1} \rho_n\ , \quad n \geq 0,$$

where the *server utilization* $\rho := \lambda/(s\mu) < 1$, and

$$\rho_n := \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n, & \text{if } 0 \leq n \leq s, \\ \frac{s^s}{s!}\rho^n, & \text{if } n \geq s+1. \end{cases}$$

- $L_Q = \sum_{n=s}^{\infty}(n-s)P_n = \sum_{n=s}^{\infty}(n-s)P_0\rho_n = \sum_{k=0}^{\infty} kP_0\rho_{s+k}$
  $= \sum_{k=1}^{\infty} kP_0\rho_s\rho^k = \sum_{k=1}^{\infty} kP_s\rho^k = \frac{P_s\rho}{(1-\rho)^2}$.
- Using Little's Law, $W_Q = L_Q/\lambda = \frac{1}{\lambda}\frac{P_s\rho}{(1-\rho)^2} = \frac{P_s}{s\mu(1-\rho)^2}$.
- $W = W_Q + \mathbb{E}[\text{service time}] = \frac{P_s}{s\mu(1-\rho)^2} + \frac{1}{\mu}$.
- Using Little's Law,
  $L = \lambda W = \lambda(W_Q + \frac{1}{\mu}) = L_Q + \frac{\lambda}{\mu} = \frac{P_s\rho}{(1-\rho)^2} + \frac{\lambda}{\mu}$.
- Due to unlimited capacity, arrival rate $=$ entering rate, so the time average (W, W_Q) is based on all customers.
- As $\rho \to 1$, all L, W, L_Q and W_Q tend to ∞.

- By letting $s \to \infty$ we get the $M/M/\infty$ queue as a limiting case of the $M/M/s$ queue.
- Note: $M/M/\infty$ queue is always stable! (The *server utilization* is always 0.)
- All the measures can be obtained by letting $s \to \infty$ for those in the case of $M/M/s$ queue.†
- Or, one can still derive P_n via the state space diagram:



† Use the Taylor expansion (泰勒展开): $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbb{R}$.

Theorem 6 (Limiting Distribution of $M/M/\infty$ Queue)

For an $M/M/\infty$ queue, its limiting (steady-state) distribution is given by

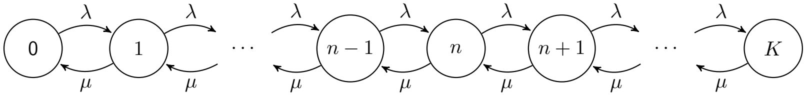
$$P_n = e^{-\lambda/\mu}\frac{(\lambda/\mu)^n}{n!}, \quad n \geq 0.$$

- In steady state, the number of customers in an $M/M/\infty$ station $\sim \mathrm{Poisson}(\lambda/\mu)$.
- Hence, $L = \sum_{n=0}^{\infty} nP_n = \mathbb{E}\left[\text{Poisson RV with mean } \frac{\lambda}{\mu}\right] = \frac{\lambda}{\mu}$.
- Using Little's Law, $W = L/\lambda = \frac{1}{\mu}$.
- $L_Q = 0$, $W_Q = 0$.

- $M/M/1/K$ Queue†
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with $\mathrm{Exp}(\mu)$ distribution.
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is K, $K \geq 1$, i.e., the maximal number of customers waiting in queue + customers in server $\leq K$.
  - A customer who finds the station is full (K customers there) leaves immediately (lost).
  - The entering rate, denoted as λ_e, is smaller than the arrival rate λ.
  - It is always stable (due to the finite capacity).
- In steady state
  - $\mathbb{P}(\text{station is full}) = P_K$.
  - Entering rate $\lambda_e = \lambda(1 - P_K)$.

† $M/M/1/K$ Queue $\subset$ *Birth and Death Process with Finite Capacity* $\subset$ Continuous-Time Markov Chain.

- The state space diagram is as follows:

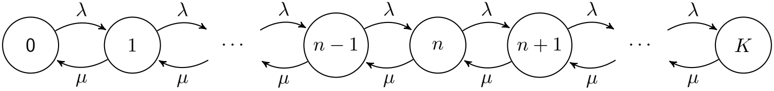


Theorem 7 (Limiting Distribution of $M/M/1/K$ Queue)

For an $M/M/1/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \begin{cases} \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}, & \text{if } \rho \neq 1, \\ \frac{1}{K+1}, & \text{if } \rho = 1, \end{cases} \qquad 0 \leq n \leq K,$$

where $\rho := \lambda/\mu$. (ρ is NOT the *server utilization*!)



Proof. Due to Observations 1 & 2,

State	Rate Process Leaves		Rate Process Enters
0	$P_0\lambda$	$=$	$P_1\mu$
$n,\ 1 \le n \le K-1$	$P_n(\mu+\lambda)$	$=$	$P_{n-1}\lambda + P_{n+1}\mu$
K	$P_K\mu$	$=$	$P_{K-1}\lambda$

Rewriting these equations gives

$$\begin{aligned} P_0\lambda &= P_1\mu, \\ P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu), \quad 1 \le n \le K-1, \\ P_K\mu &= P_{K-1}\lambda. \end{aligned}$$

Or, equivalently,

$$
\begin{aligned}
P_0\lambda &= P_1\mu, \\
P_1\lambda &= P_2\mu + (P_0\lambda - P_1\mu) = P_2\mu, \\
P_2\lambda &= P_3\mu + (P_1\lambda - P_2\mu) = P_3\mu, \\
P_n\lambda &= P_{n+1}\mu + (P_{n-1}\lambda - P_n\mu) = P_{n+1}\mu, \quad 1 \le n \le K-2, \\
P_{K-1}\lambda &= P_K\mu.
\end{aligned}
$$

Let $\rho := \lambda/\mu$, solving in terms of P_0 yields

$$
\begin{aligned}
P_1 &= P_0\rho, \\
P_2 &= P_1\rho = P_0\rho^2, \\
P_n &= P_{n-1}\rho = P_0\rho^n, \quad 1 \le n \le K.
\end{aligned}
$$

Since $1 = \Sigma_{n=0}^{K} P_n = P_0 \Sigma_{n=0}^{K} \rho^n = \begin{cases} P_0 \frac{1-\rho^{K+1}}{1-\rho}, & \text{if } \rho \neq 1, \\ P_0(K+1), & \text{if } \rho = 1, \end{cases}$ we have,

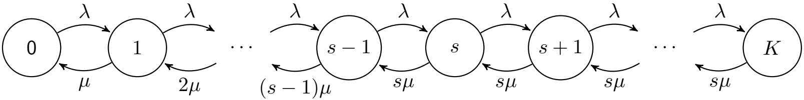
if $\rho \neq 1$, $\quad P_0 = \frac{1-\rho}{1-\rho^{K+1}}$, and $\quad P_n = \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}$, $\quad 1 \le n \le K$;

if $\rho = 1$, $\quad P_0 = \frac{1}{K+1}$, and $\quad P_n = \frac{1}{K+1}$, $\quad 1 \le n \le K$. $\blacksquare$

- If $\rho \neq 1$,
$$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{(1-\rho)\rho^n}{1-\rho^{K+1}} = \frac{1-\rho}{1-\rho^{K+1}}\sum_{n=0}^{K} n\rho^n$$
$$= \frac{1-\rho}{1-\rho^{K+1}}\frac{\rho-(K+1)\rho^{K+1}+K\rho^{K+2}}{(1-\rho)^2} = \frac{\rho}{1-\rho}\frac{1-(K+1)\rho^{K}+K\rho^{K+1}}{1-\rho^{K+1}}.$$
- If $\rho = 1$,
$$L = \sum_{n=0}^{K} nP_n = \sum_{n=0}^{K} n\frac{1}{K+1} = \frac{1}{K+1}\frac{(K+1)K}{2} = \frac{K}{2}.$$
- $\mathbb{P}(\text{station is full}) = P_K$.
- Entering rate $\lambda_e = \lambda(1-P_K)$.
- The *server utilization* $= \lambda_e/\mu = \rho(1-P_K)$.
- As $\rho \to \infty$, $L \to K$, $1-P_K \to 0$, $\rho(1-P_K) \to 1$.

- For those entered the station
  - The expected sojourn time $W = L/\lambda_e = \frac{L}{\lambda(1-P_K)}$.
  - The expected waiting time $W_Q = W - \frac{1}{\mu} = \frac{L}{\lambda(1-P_K)} - \frac{1}{\mu}$.
- For ALL the arrivals (those who are lost have 0 sojourn time and waiting time)
  - The expected sojourn time $W' = (1-P_K)W + 0 = \frac{L}{\lambda}$.
  - The expected waiting time $W'_Q = (1-P_K)W_Q + 0 = \frac{L}{\lambda} - \frac{1-P_K}{\mu}$.
- The expected queue length $L_Q = \lambda_e W_Q = L - \rho(1-P_K)$,
  or, $= \lambda W'_Q = L - \rho(1-P_K)$.
- As $\rho \to \infty$, $1-P_K \to 0$, $\rho(1-P_K) \to 1$, $L \to K$, $L_Q \to K-1$.
  - If μ is fixed and $\lambda \to \infty$:
    $\lambda(1-P_K) \to \mu$, $W \to \frac{K}{\mu}$, $W_Q \to \frac{K-1}{\mu}$, $W' \to 0$, $W'_Q \to 0$.
  - If λ is fixed and $\mu \to 0$:
    $\frac{1}{\mu}(1-P_K) \to \frac{1}{\lambda}$, $W \to \infty$, $W_Q \to \infty$, $W' \to \frac{K}{\lambda}$, $W'_Q \to \frac{K-1}{\lambda}$.

- $M/M/s/K$ queue† is a generalized version of $M/M/1/K$ queue. $(K \geq s)$
- The state space diagram is as follows:



- Let $s = 1$, it becomes the $M/M/1/K$ queue.
- Let $s = K$, it becomes the $M/M/K/K$ queue.
- There is no $M/M/\infty/K$ queue!

† $M/M/1/K$ Queue $\subset$ $M/M/s/K$ Queue $\subset$ *Birth and Death Process with Finite Capacity* $\subset$ CTMC.

Theorem 8 (Limiting Distribution of $M/M/s/K$ Queue)

For an $M/M/s/K$ queue, its limiting (steady-state) distribution is given by

$$P_n = \left[\sum_{i=0}^{s} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i + \varrho\right]^{-1} \rho_n\ , \quad 0 \le n \le K,$$

where $\rho := \lambda/(s\mu)$, (ρ is NOT the *server utilization*!) and

$$\varrho := \begin{cases} \frac{s^s}{s!}\frac{\rho^{s+1}(1-\rho^{K-s})}{1-\rho}, & \text{if } \rho \neq 1, \\ \frac{s^s}{s!}(K-s), & \text{if } \rho = 1, \end{cases}$$

and

$$\rho_n := \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n, & \text{if } 0 \le n \le s, \\ \frac{s^s}{s!}\rho^n, & \text{if } s+1 \le n \le K,\ K \ge s+1. \end{cases}$$

- The *server utilization* $= \lambda_e/(s\mu) = \rho(1 - P_K)$.

- $M/G/1$ Queue[†]
  - Customers arrive according to a Poisson process with rate λ.
  - The service times are iid random variables with **arbitrary** distribution (mean: $\frac{1}{\mu}$, variance: σ^2).
  - The customers are served in an FCFS fashion by a *single* server.
  - The capacity is unlimited, i.e., waiting space is unlimited.
  - $M/G/1$ queue is stable **if and only if** $\lambda < \mu$.

- Let $m^2 := \left(\frac{1}{\mu}\right)^2 + \sigma^2$, and the *server utilization* $\rho := \lambda/\mu < 1$.
  - $\mathbb{P}(\text{the server is idle}) = 1 - \rho$.
  - $W_Q = \frac{\lambda m^2}{2(1-\rho)}$.
  - $L_Q = \lambda W_Q = \frac{\lambda^2 m^2}{2(1-\rho)}$.
  - $W = W_Q + \frac{1}{\mu} = \frac{\lambda m^2}{2(1-\rho)} + \frac{1}{\mu}$.
  - $L = \lambda W = L_Q + \lambda/\mu = \frac{\lambda^2 m^2}{2(1-\rho)} + \rho$.

- For $M/G/\infty$, the measures are the same as those in $M/M/\infty$.

[†] $M/G/1$ queue has an embedded discrete-time Markov chain.

- Queueing Network (multiple-station queueing system)
  - Customers can move from one station to another (for different service), before leaving the system.

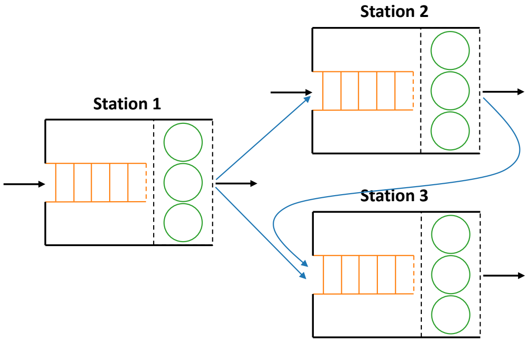


Figure: Illustration of Queueing Networks

- Jackson Queueing Network (first identified by Jackson (1963))[†]
  1. The network has J single-station queues.
  2. The jth station has s_j servers and a *single* queue.
  3. There is unlimited waiting space at each station (infinite capacity).
  4. Customers arrive at station j from outside according to a Poisson process with rate λ_j; all arrival processes are independent of each other.
  5. The service times at station j are iid random variables with $\mathrm{Exp}(\mu_j)$ distribution.
  6. Customers finishing service at station i join the queue (if any) at station j with **routing probability** p_{ij}, or leave the network with probability p_{i0}, independently of each other.
  7. A customer finishing service may be routed to the same station (i.e., re-enter).

[†] Jackson network is an J-dimensional continuous-time Markov chain.

- The routing probabilities p_{ij} can be put in a matrix form as follows:

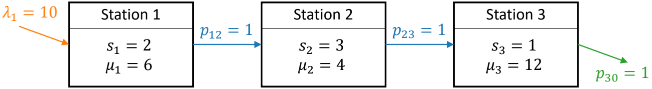
$$\boldsymbol{P} := \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1J} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2J} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{J1} & p_{J2} & p_{J3} & \cdots & p_{JJ} \end{bmatrix}.$$

- The matrix $\boldsymbol{P}$ is called the **routing matrix**.

- Since a customer leaving station i either joints some other station, or leaves, we must have

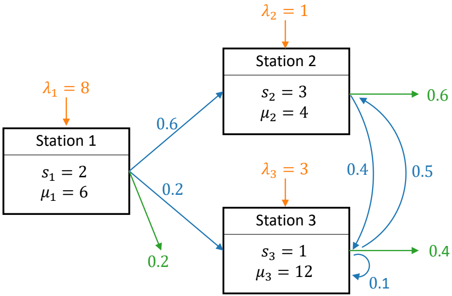
$$\sum_{j=1}^{J} p_{ij} + p_{i0} = 1, \quad 1 \le i \le J.$$

- Example 1: Tandem Queue



$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Example 2: General Network



$$\boldsymbol{P} = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

- Recall that customers arrive at station j from outside with rate λ_j.
- Let b_j be the rate of internal arrivals to station j.
- Then the total arrival rate to station j, denoted as a_j, is given by
$$a_j = \lambda_j + b_j, \quad 1 \leq j \leq J.$$
- If the stations are all **stable**
  - The departure rate of customers from station i will be the same as the total arrival rate to station i, namely, a_i.
  - The arrival rate of internal customers from station i to station j is $a_i p_{ij}$.
- Hence, $b_j = \sum_{i=1}^{J} a_i p_{ij}, \quad 1 \leq j \leq J$.
- Substituting in the pervious equation, we get the **traffic equations**:
$$a_j = \lambda_j + \sum_{i=1}^{J} a_i p_{ij}, \quad 1 \leq j \leq J.$$

- Let $\boldsymbol{a}^\intercal = [a_1 \ a_2 \ \cdots \ a_J]$ and $\boldsymbol{\lambda}^\intercal = [\lambda_1 \ \lambda_2 \ \cdots \ \lambda_J]$, the traffic equations can be written in matrix form as
$$\boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal + \boldsymbol{a}^\intercal \boldsymbol{P},$$
or
$$\boldsymbol{a}^\intercal (\boldsymbol{I} - \boldsymbol{P}) = \boldsymbol{\lambda}^\intercal,$$
where $\boldsymbol{I}$ is the $J \times J$ identity matrix.

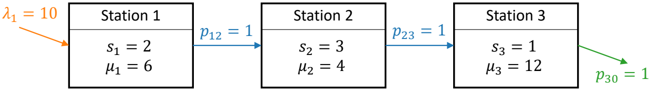
- Suppose the matrix $\boldsymbol{I} - \boldsymbol{P}$ is invertible, the above equation has a unique solution given by
$$\boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal (\boldsymbol{I} - \boldsymbol{P})^{-1}.$$

- The next theorem states the stability condition for Jackson networks in terms of the above solution.

Theorem 9 (Stability of Jackson Networks)

A Jackson network with external arrival rate vector $\boldsymbol{\lambda}$ and routing matrix $\boldsymbol{P}$ is stable if:
(1) $\boldsymbol{I} - \boldsymbol{P}$ is invertible; and
(2) $a_i < s_i\mu_i$ for all $i = 1, 2, \ldots, J$, where a_i is given by the traffic equations.

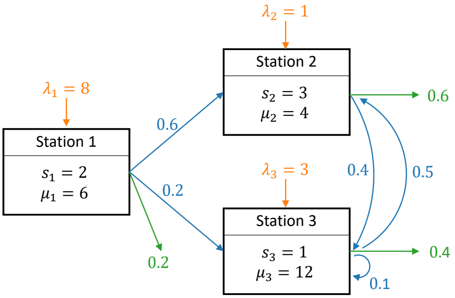
- Example 1: Tandem Queue



$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal(\boldsymbol{I} - \boldsymbol{P})^{-1} = [10\ 10\ 10].$$

Stable.

- Example 2: General Network



$$\boldsymbol{P} = \begin{bmatrix} 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 \\ 0 & 0.5 & 0.1 \end{bmatrix}.$$

$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal(\boldsymbol{I} - \boldsymbol{P})^{-1} = [8\ 10.7\ 9.9] \Rightarrow \text{Stable.}$$

If λ_2 is increased to 4,

$$\boldsymbol{\lambda} = \begin{bmatrix} 8 \\ 4 \\ 3 \end{bmatrix}, \quad \boldsymbol{a}^\intercal = \boldsymbol{\lambda}^\intercal(\boldsymbol{I} - \boldsymbol{P})^{-1} = [8\ 14.6\ 11.6] \Rightarrow \text{Unstable.}$$

- Let $L_j(t)$ be the number of customers in the jth station in a Jackson network at time t.
- Then the state of the network at time t is given by $[L_1(t), L_2(t), \ldots, L_J(t)]$.
- When the Jackson network is stable, the limiting distribution of the sate of the network is

$$\begin{aligned} &P(n_1, n_2, \ldots, n_J) \\ &\qquad = \lim_{t\to\infty} \mathbb{P}\{L_1(t) = n_1, L_2(t) = n_2, \ldots, L_J(t) = n_J\}. \end{aligned}$$

- It is a joint probability.

Theorem 10 (Limiting Distribution of Jackson Network)

For a stable Jackson network, its limiting (steady-state) distribution is given by

$$P(n_1, n_2, \ldots, n_J) = P_1(n_1)P_2(n_2)\cdots P_J(n_J),$$

for $n_j = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots, J$, where $P_j(n)$ is the limiting probability that there are n customers in an $M/M/s_j$ queue with arrival rate a_j and service rate μ_j.

- The limiting **joint** distribution of $[L_1(t), \ldots, L_J(t)]$ is a **product** of the limiting **marginal** distribution of $L_j(t)$, $j = 1, \ldots, J$.
  $\Rightarrow$ Limiting behavior of all stations are independent of each other.
- The limiting distribution of station j is the same as that in an **isolated** $M/M/s_j$ queue with arrival rate a_j and service rate μ_j. (a_j's are solved from the **traffic equations**.)